Chapter 1

Coupled Nonlinear Shrödinger-KdV Equations

1.1 Introduction

In this chapter we will study the coupled nonlinear Shrödinger-KdV equations (CSKdV) and the exact solution of them. So we will see that this equations has conserved quantities. We will present the solution of the block tridiagonal system, penta-diagonal system and block penta-diagonal system. Fixed point method for solving the nonlinear system will be given.

Exact solutions for coupled nonlinear systems are discussed by many authers [1],[10],[21],[22],[28]. Also the numerical solution for coupled nonlinear Shrödinger-KdV are studied by many authers and very rich research subject [2]-[5],[11],[12]-[19],[23],[24]. Finite element solution of the CSKdV are discussed by [6],[7].

1.2 Coupled Nonlinear Shrödinger-KdV Equations

Nonlinear phenomena play a crucial role in a variety of scientific fields, especially in fluid mechanics, solid state physics, plasma physics, plasma waves and chemical physics. The coupled nonlinear Shrödinger-KdV equations [6],[7]

$$i\epsilon u_t + \frac{3}{2}u_{xx} - \frac{1}{2}uv = 0 \; , \tag{1.1}$$

$$v_t + \frac{1}{2}v_{xxx} + \frac{1}{2}\left(|u|^2 + v^2\right)_x = 0 \; , \tag{1.2}$$

$$x_L < x < x_R, \quad t > 0$$

with the initial conditions

$$u(x,0) = u_0(x) \; , \qquad v(x,0) = v_0(x)$$

and boundary conditions

$$u(x_l,t) = u(x_r,t) = 0 \; , \qquad v(x_l,t) = v(x_r,t) = 0 \tag{1.3}$$

have been used extensively to model nonlinear dynamics of one-dimensional Langmuir and ion-acoustic waves in a system of coordinates moving at the ion-acoustic speed. Here $u(x,t)$ is a complex function describing electric field of Langmuir oscillations and $v(x,t)$ is real function describing low-frequency density perturbation. $\epsilon > 0$ is a constant [7]. The exact solution of coupled Shrödinger-KdV equations (1.1) and (1.2) is

$$u(x,t) = -\frac{6}{5}\sqrt{3}\alpha\frac{\tanh\xi}{\cosh\xi}\exp\left\{i\alpha\left[\left(\frac{3}{20\epsilon} - \frac{\epsilon\alpha}{6}\right)t - \frac{\epsilon x}{3}\right]\right\} \; , \tag{1.4}$$

$$v(x,t) = -\frac{9}{5}\alpha\frac{1}{\cosh^2\xi} \tag{1.5}$$

where $\xi = \sqrt{\frac{\alpha}{10}}(x + \alpha t)$, and α is a free positive parameter [6],[7].

To avoid complex computation, we assume [12]-[19]

$$u(x,t) = u_1(x,t) + iu_2(x,t), \; i^2 = -1 , \quad (1.6)$$

$$v(x,t) = u_3(x,t) \quad (1.7)$$

where $u_1(x,t), u_2(x,t)$ and $u_3(x,t)$ are real functions.

By making use of (1.6) and (1.7), the CSKdV equations will be reduced to the coupled system

$$\epsilon (u_1)_t + \frac{3}{2}(u_2)_{xx} - \frac{1}{2}u_2u_3 = 0 , \quad (1.8)$$

$$\epsilon (u_2)_t - \frac{3}{2}(u_1)_{xx} + \frac{1}{2}u_1u_3 = 0 , \quad (1.9)$$

$$(u_3)_t + \frac{1}{2}(u_3)_{xxx} + \frac{1}{2}\left(u_1^2 + u_2^2 + u_3^2\right)_x = 0 . \quad (1.10)$$

System (1.8) - (1.10) is nonlinear [10].

1.3 Conservation Laws

Theorem:

The coupled nonlinear Shrödinger-KdV equations (1.1) and (1.2) has the conserved quantities[6]:

i) The number of plasmons:

$$I_1 = \int_{-\infty}^{\infty} |u|^2 \, dx \quad (1.11)$$

ii) The number of particles:

$$I_2 = \int_{-\infty}^{\infty} u_3 \, dx \quad (1.12)$$

iii) The energy of the oscillations:

$$I_3 = \int_{-\infty}^{\infty} \left[3|u_x|^2 + u_3|u|^2 + \frac{1}{3}u_3^3 - \frac{1}{2}(u_{3x})^2 \right] dx \quad (1.13)$$

Proof:

i) The number of plasmons:

To prove (1.11), we multiply equation (1.8) and (1.9) by u_1 and u_2, respectively, then by adding the resulting equations to get

$$\epsilon\frac{\partial}{\partial t}\left(u_1^2+u_2^2\right)+\frac{3}{2}\frac{\partial}{\partial x}\left(u_1u_{2x}-u_2u_{1x}\right)=0$$

Integrate both sides of the previous equation with respect to x, to get

$$\epsilon\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\left(u_1^2+u_2^2\right)dx+\frac{3}{2}\left[u_1u_{2x}-u_2u_{1x}\right]_{-\infty}^{\infty}=0 \tag{1.14}$$

Assuming vanishing boundary conditions, the last term of equation (1.14) is zero, so,

$$\epsilon\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\left(u_1^2+u_2^2\right)dx=0$$

and hence the first conserved quantity (1.11) is obtained. The value of I_1, using the exact solution is

$$I_1=\int_{-\infty}^{\infty}|u|^2\,dx=\frac{72}{25}\sqrt{10\alpha^3}$$

ii) The number of particles:

To prove the second conserved quantity (1.12), we integrate both sides of equation (1.10) with respect to x, this will gives us

$$\int_{-\infty}^{\infty}(u_3)_t\,dx+\frac{1}{2}\int_{-\infty}^{\infty}(u_3)_{xxx}\,dx\ +\ \frac{1}{2}\int_{-\infty}^{\infty}\left(u_1^2+u_2^2+u_3^2\right)_x dx=0$$

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}u_3dx+\frac{1}{2}\left[(u_3)_{xx}\ +\ \left(u_1^2+u_2^2+u_3^2\right)_x\right]_{-\infty}^{\infty}=0\ , \tag{1.15}$$

The second term in (1.15) will vanish due to the vanishing boundary conditions and this will lead us to

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}u_3dx=0\ ,$$

and hence

$$\int_{-\infty}^{\infty} u_3 dx = \text{constant}$$

Using the exact solution we can easily find the exact value of I_2

$$I_2 = \int_{-\infty}^{\infty} u_3 dx = -\frac{18}{5}\sqrt{10\alpha}$$

iii) The energy of the oscillations:

To prove (1.13), we multiply equation (1.8) and (1.9) by $2u_1u_3$ and $2u_2u_3$, respectively, then we get

$$2\epsilon u_1 u_3 (u_1)_t + 3u_1u_3 (u_2)_{xx} - u_1u_2u_3^2 = 0 , \tag{1.16}$$

$$2\epsilon u_2 u_3 (u_2)_t - 3u_2u_3 (u_1)_{xx} + u_2u_1u_3^2 = 0 , \tag{1.17}$$

after that, we differentiate equation (1.8), (1.9) and (1.10) with respect to x, then multiply result by $6u_{1x}$, $6u_{2x}$ and u_{3x}, respectively, then we get

$$6\epsilon u_{1x} (u_1)_{tx} + 9u_{1x} (u_2)_{xxx} - 3u_{1x} (u_{2x}u_3 + u_2u_{3x}) = 0 , \tag{1.18}$$

$$6\epsilon u_{2x} (u_2)_{tx} - 9u_{2x} (u_1)_{xxx} + 3u_{2x} (u_{1x}u_3 + u_1u_{3x}) = 0 , \tag{1.19}$$

$$u_{3x} (u_3)_{tx} + \frac{1}{2} u_{3x} (u_3)_{xxxx} + \frac{1}{2} u_{3x} \left(u_1^2 + u_2^2 + u_3^2\right)_{xx} = 0 , \tag{1.20}$$

after that, we multiply equation (1.10) by u_1^2, u_2^2 and by u_3^2

$$u_1^2 (u_3)_t + \frac{1}{2} u_1^2 (u_3)_{xxx} + \frac{1}{2} u_1^2 \left(u_1^2 + u_2^2 + u_3^2\right)_x = 0 , \tag{1.21}$$

$$u_2^2 (u_3)_t + \frac{1}{2} u_2^2 (u_3)_{xxx} + \frac{1}{2} u_2^2 \left(u_1^2 + u_2^2 + u_3^2\right)_x = 0 , \tag{1.22}$$

$$u_3^2 (u_3)_t + \frac{1}{2} u_3^2 (u_3)_{xxx} + \frac{1}{2} u_3^2 \left(u_1^2 + u_2^2 + u_3^2\right)_x = 0 . \tag{1.23}$$

Finally, adding equations (1.16) - (1.23), so that the resulting equation can be rewritten as:

$$\frac{\partial}{\partial t}\left[3\left|u_x\right|^2+u_3\left|u\right|^2+\frac{1}{3}u_3^3-\frac{1}{2}\left(u_{3x}\right)^2\right]+\frac{\partial}{\partial x}[u_3\left(u_1u_{2x}-u_{1x}u_2\right)$$
$$+\left(u_{1x}u_{2xx}-u_{1xx}u_{2x}\right)+\left(u_1^2+u_2^2+u_3^2\right)^2+\left(u_{3x}u_{3xxx}-\frac{1}{2}u_{3xx}^2\right)$$
$$+(u_{3x}\left(u_1^2+u_2^2+u_3^2\right)-u_{3xx}\left(u_1^2+u_2^2+u_3^2\right))]=0\ . \tag{1.24}$$

We integrate both sides of equation (1.24) with respect to x, to get

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\left[3\left|u_x\right|^2+u_3\left|u\right|^2+\frac{1}{3}u_3^3-\frac{1}{2}\left(u_{3x}\right)^2\right]dx+\frac{\partial}{\partial x}\int_{-\infty}^{\infty}[u_3\left(u_1u_{2x}-u_{1x}u_2\right)$$
$$+\left(u_{1x}u_{2xx}-u_{1xx}u_{2x}\right)+\left(u_1^2+u_2^2+u_3^2\right)^2+\left(u_{3x}u_{3xxx}-\frac{1}{2}u_{3xx}^2\right)$$
$$+\left(u_{3x}\left(u_1^2+u_2^2+u_3^2\right)\ -u_{3xx}\left(u_1^2+u_2^2+u_3^2\right)\right)]dx=0\ .$$

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\left[3\left|u_x\right|^2+u_3\left|u\right|^2+\frac{1}{3}u_3^3-\frac{1}{2}\left(u_{3x}\right)^2\right]dx+[u_3\left(u_1u_{2x}-u_{1x}u_2\right)$$
$$+\left(u_{1x}u_{2xx}-u_{1xx}u_{2x}\right)+\left(u_1^2+u_2^2+u_3^2\right)^2+\left(u_{3x}u_{3xxx}-\frac{1}{2}u_{3xx}^2\right)$$
$$+\left(u_{3x}\left(u_1^2+u_2^2+u_3^2\right)-u_{3xx}\left(u_1^2+u_2^2+u_3^2\right)\right)]_{-\infty}^{\infty}=0 \tag{1.25}$$

where

$$[u_3\left(u_1u_{2x}-u_{1x}u_2\right)+\left(u_{1x}u_{2xx}-u_{1xx}u_{2x}\right)+\left(u_1^2+u_2^2+u_3^2\right)^2$$
$$+\left(u_{3x}u_{3xxx}-\frac{1}{2}u_{3xx}^2\right)+\left(u_{3x}\left(u_1^2+u_2^2+u_3^2\right)-u_{3xx}\left(u_1^2+u_2^2+u_3^2\right)\right)]_{-\infty}^{\infty}\longrightarrow 0$$

by using the boundary conditions, then equation (1.25) becomes:

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\left[3\left|u_x\right|^2+u_3\left|u\right|^2+\frac{1}{3}u_3^3-\frac{1}{2}\left(u_{3x}\right)^2\right]dx=0$$

and this gives$\int_{-\infty}^{\infty}\left[3\left|u_x\right|^2+u_3\left|u\right|^2+\frac{1}{3}u_3^3-\frac{1}{2}\left(u_{3x}\right)^2\right]$ =constant

1.4 Solution of Block Tridiagonal System

In our numerical calculations, we need the solution of block tridiagonal system. Crout's method is used to solve this system, and this method can be described as follows [2],[5]:

Consider the block tridiagonal system

$$A_i x_{i-1} + B_i x_i + C_i x_{i+1} = F_i \ , \qquad \text{for } i = 1, 2, ..., n$$

where

$$A_1 = C_n = 0$$

We can write this system in a matrix vector form as:

$$G\mathbf{x} = \mathbf{F} \tag{1.26}$$

$$\begin{bmatrix} B_1 & C_1 & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ A_2 & B_2 & C_2 & & & \vdots \\ \mathbf{0} & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \mathbf{0} \\ \vdots & & & A_{n-1} & B_{n-1} & C_{n-1} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & A_n & B_n \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \vdots \\ \mathbf{x}_{n-1} \\ \mathbf{x}_n \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ \vdots \\ F_{n-1} \\ F_n \end{bmatrix}$$

here each A_i is an $(m_i \times m_{i-1})$ matrix, each B_i is an $(m_i \times m_i)$ matrix and each C_i is an $(m_i \times m_{i+1})$ matrix for some collection of positive integers $m_1, m_2, ..., m_n$. and so x_i and F_i are $(m \times 1)$ column subvectors and $\mathbf{0}$ denotes the $(m \times m)$ zero matrix.

To solve the block tridiagonal system we factor the matrix G in equation (1.26) as

$$G = LU \tag{1.27}$$

where

$$L = \begin{bmatrix} L_1 & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ A_2 & L_2 & & & \vdots \\ \mathbf{0} & \ddots & \ddots & & \vdots \\ \vdots & & A_{n-1} & L_{n-1} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & A_n & L_n \end{bmatrix} ,$$

$$U = \begin{bmatrix} I_1 & U_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & I_2 & U_2 & & \vdots \\ \vdots & & \ddots & \ddots & \mathbf{0} \\ \vdots & & & I_{n-1} & U_{n-1} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & I_n \end{bmatrix}$$

where L and U are lower and upper block triangular matrices respectively, and each L_i is $(m_i \times m_i)$ matrix, each U_i is $(m_i \times m_{i+1})$ matrix and each I_i is $(m_i \times m_i)$ identity matrix.

Now multiply the right hand side of equation (1.27) and equate both sides of equation (1.27). We can easily find the unknown elements $\{L_i\}_{i=1}^{n}$ and $\{U_i\}_{i=1}^{n-1}$ in the following manner

$$\begin{aligned} L_1 &= B_1 \ , \\ U_1 &= B_1^{-1} C_1 \ , \end{aligned}$$

$$\begin{aligned} L_i &= B_i - A_i U_{i-1} \ , \\ U_i &= L_i^{-1} C_i \end{aligned}$$

for $i = 2, 3, ...n-1$, and

$$L_n = B_n - A_n U_{n-1}$$

Now the system (1.26) can be written as

$$LU\mathbf{x} = \mathbf{F} \ . \tag{1.28}$$

Now by assuming

$$U\mathbf{x} = \mathbf{y} \ . \tag{1.29}$$

Equation (1.28) will be reduced to

$$L\mathbf{y} = \mathbf{F} \ . \tag{1.30}$$